Help Notes: Induction

27 octobre 2017

Induction principle

Dominoes are Falling Down

If you queued ten thousand dominoes on a very long table and you want to let them all fall just by letting the first domino fall, then how would you queue it ?

The best idea probably is to queue them such that :

1. When the first domino falls, it will hit the second domino.
2. Make sure that each domino will hit the domino next to it and that each hit domino will fall.

If conditions (1) and (2) are satisfied, then all the dominoes will fall.

What about the math ?

Mathematical Induction states that

Theorem 0.1. *If P is a condition and $P(1)$ is true, and for a natural number n , if $P(n)$ then $P(n+1)$ is true, then P is true for every positive integer.*

For example, we want to add the first n natural numbers, we may observe that

$$1 = 1$$

If we continue, we might observe that

$$1 + 2 = \frac{(2)(3)}{2}$$

and

$$1 + 2 + 3 = \frac{(3)(4)}{2}$$

and we might be tempted to guess that

$$1 + 2 + 3 + \ldots + 99 + 100 = \frac{(100)(101)}{2}.$$

You may want to try a few more cases.

Note that the ? sign means all the way through 100. This means that we want to add all the integers from 1 through 100.

Our guess above is actually correct (Why ?), but we do not know if this formula or strategy always works for all positive numbers. For instance, how do we know if the formula works if the largest number is 500, or 2000, or $1,000,000$. Of course, we cannot enumerate all the counting numbers, so we need to have a justification or proof of our conjecture above. The strategy used for proving such conjectures is called proof by mathematical induction.

In mathematical induction, if our condition is true for the natural number $n = 1$, and once it is true for any natural number $n = k$, it is also true for $n = k + 1$, then the condition is true for all positive integers. If it is a little unclear, take this analogy :

Suppose we have an infinite number of dominoes queued, and once we push the first domino (this is our $n = 1$), it hits the next domino and the second domino falls. Now, if for each if the hit domino falls (this is our $n = k$) and hits the next domino and again that domino (this is our $n = k + 1$), then we are sure that all the standing dominoes will fall no matter how long their queue is.

Now, for our problem, we want to show that the sum of the numbers from 1 to any number say n, or written as

$$1 + 2 + 3 + \ldots + n = \frac{(n)(n+1)}{2}$$

Note that n is any positive integer. We check if the condition is true for n = 1. That is the initialization step on the induction proof

$$1 = \frac{(1)(1+1)}{2}$$

Yes, that is true. The right hand side of the equal sign, $1(1 + 1)/2$ indeed equals 1.

Assuming $n = k$ is true, that is

$$1 + 2 + 3 + \ldots + k = \frac{(k)(k+1)}{2}.$$

That is the Induction assumption (this you are assuming to be true, and you need to use this assumption sometime in your reasoning if you do not use it then it is that you did not need an induction.)

We must show that n = k+1 is also true. That is we must show that

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

In order to prove an equality we need to prove that both side are equal, thus either choose a side and prove via a succession of equality that it is the same as the right side or work on both sides **separately** and do a succession of equalities to reach the other side do not prove an equality supposing the equality!! With practice you will get intuition on what to do and what is best. Here the best is to start from the left side and reach the right side.

Note that since $n = k + 1$, we just replace all n by $k + 1$ in the right hand of side of the equation.

So,

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k)(k+1)}{2} + (k+1)$$

Simplifying the right hand side, we have

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2},$$

and the property is true for $n = k + 1$

We have then proven using mathematical induction that for any $n \leqslant 1$,

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Note : we have proven something for a finite amount of integer in two steps. It is great!!!